# A CLASSIFICATION OF NON-HERMITIAN RANDOM MATRICES.

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**Abstract** We present a classification of non-hermitian random matrices based on implementing commuting discrete symmetries. It contains 43 classes. This generalizes the classification of hermitian random matrices due to Altland-Zirnbauer and it also extends the Ginibre ensembles of non-hermitian matrices [1].

Random matrix theory originates from the work of Wigner and Dyson on random hamiltonians [2]. Since then it has been applied to a large variety of problems ranging from enumerative topology, combinatorics, to localization phenomena, fluctuating surfaces, integrable or chaotic systems, etc... Non-hermitian random matrices also have applications to interesting quantum problems such as open choatic scattering, dissipative quantum maps, non-hermitian localization, etc... See e.g. ref.[6] for an introduction. The aim of this short note is to extend the Dyson [2] and Altland-Zirnbauer [4] classifications of hermitian random matrix ensembles to the non-hermitian ones.

## 1. What are the rules?

As usual, random matrix ensembles are constructed by selecting classes of matrices with specified properties under discrete symmetries [2, 3]. To define these ensembles we have to specify (i) what are the discrete symmetries, (ii) what are the equivalence relations among the matrices, and (iii) what are the probablility measures for each class.

(i) What are these discrete symmetries.
Let $h$ denote a complex matrix. We demand that the transformations specifying random matrix classes are involutions — their actions are of order two. So we consider the following set of symmetries:

$$\begin{aligned} &\text{C sym.} &:& \quad h = \epsilon_c\, c\, h^T\, c^{-1}, \quad c^T c^{-1} = \pm \mathbf{1} &(1)\\ &\text{P sym.} &:& \quad h = -p\, h\, p^{-1}, \quad p^2 = \mathbf{1} &(2)\\ &\text{Q sym.} &:& \quad h = q\, h^\dagger\, q^{-1}, \quad q^\dagger q^{-1} = \mathbf{1} &(3)\\ &\text{K sym.} &:& \quad h = k\, h^*\, k^{-1}, \quad kk^* = \pm \mathbf{1} &(4) \end{aligned}$$

$h^T$ denotes the transposed matrix of $h$, $h^*$ its complex conjugate and $h^\dagger$ its hermitian conjugate. The factor $\epsilon_c$ is just a sign $\epsilon_c = \pm$. We could have introduced similar signs in the definitions of type $Q$ and type $K$ symmetries; however they can be removed by redefining $h \to ih$.

We also demand that these transformations are implemented by *unitary* transformations:

$$cc^\dagger = \mathbf{1},\ pp^\dagger = \mathbf{1},\ qq^\dagger = \mathbf{1},\ kk^\dagger = \mathbf{1} \tag{5}$$

In the case of hermitian matrices one refers to type $C$ symmetries as particle/hole symmetries or time reversal symmetries depending on whether $\epsilon_c = -$ or $\epsilon_c = +$ respectively. Matrices with type $P$ symmetry are said to be chiral. Both type $Q$ and type $K$ symmetries impose reality conditions on $h$ and they are redundant for hermitian matrices.

(ii) What are the equivalence relations.
We consider matrices up to *unitary* changes of basis,

$$h \to u\, h\, u^\dagger \tag{6}$$

In other words, matrices linked by unitary similarity transformations are said to be gauge equivalent. For the symmetries (1–4), this gauge equivalence translates into:

$$c \to ucu^T,\ p \to upu^{-1},\ q \to uqu^\dagger,\ k \to uku^{*-1} \tag{7}$$

The classification relies heavily on this rule and on the assumed unitary implementations of the discrete symmetries.

We shall only classify minimal classes, which by definition are those whose matrices do not commute with a fixed matrix.

(iii) What are the probability measures.
Since each of the classes we shall describe below is a subset of the space of complex matrices, the simplest probability measure $\mu(dh)$ one may choose is obtained by restriction of the gaussian one defined by

$$\mu(dh) = \mathcal{N}\, \exp(-\mathrm{Tr} hh^\dagger\,)\, dh \tag{8}$$

with $\mathcal{N}$ a normalization factor. It is invariant under the map (6).

There is of course some degree of arbitrariness in formulating these rules, in particular concerning the choice of the gauge equivalence (6). It however originates on one hand by requiring the gaussian measure (8) be invariant, and one the another hand from considering auxiliary hermitian matrices $\mathcal{H}$ obtained by doubling the vector spaces on which the matrices $h$ are acting. These doubled matrices are defined by:

$$\mathcal{H} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \tag{9}$$

They are always chiral as they anticommute with $\gamma_5 = \mathrm{diag}(1, -1)$. Any similarity transformations $h \to uhu^{-1}$ are mapped into $\mathcal{H} \to \mathcal{U}\mathcal{H}\mathcal{U}^\dagger$ with $\mathcal{U} = \mathrm{diag}(u, u^{\dagger\,-1})$. So, demanding that these transformations also act by similarity on $\mathcal{H}$ imposes $u$ to be unitary.

On $\mathcal{H}$, both type $P$ and $Q$ symmetries act as chiral transformations, $\mathcal{H} \to -\mathcal{P}\mathcal{H}\mathcal{P}^{-1}$ with $\mathcal{P} = \mathrm{diag}(P, P)$ and $\mathcal{H} \to \mathcal{Q}\mathcal{H}\mathcal{Q}^{-1}$ with $\mathcal{Q} = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}$, and $\mathcal{H}$ may be block diagonalized if $h$ is $Q$ or $P$ symmetric. Indeed, if $h$ is $Q$ symmetric then $\mathcal{Q}$ and $\mathcal{H}$ may be simultaneously diagonalized since they commute. If $h$ is $P$ symmetric, $\mathcal{H}$ commutes with the product $\gamma_5\mathcal{P}$.

Type $C$ and $K$ symmetries both act as particle/hole symmetries relating $\mathcal{H}$ to its transpose $\mathcal{H}^T$. The classification of the doubled hamiltonians $\mathcal{H}$ thus reduces to that of chiral random matrices, cf. [4]. However, the spectra of $h$ and $\mathcal{H}$ may differ significantly so that we need a finer classification involving $h$ per se.

## 2. Intrinsic definition of classes.

To specify classes we demand that the matrices belonging to a given class be invariant under one or more of the symmetries (1–4). It is important to bear in mind that when imposing two or more symmetries it is the group generated by these symmetries which is meaningful. Indeed, these groups may be presented in various ways depending on which generators one picks. For instance, if a matrix possesses both a type $P$ and a type $C$ symmetry, then it automatically has another type $C$ symmetry with $c' = pc$ and $\epsilon'_c = -\epsilon_c$.

The intrinsic classification concerns the classification of the symmetry groups generated by the transformations (1–4).

We demand, as usual, that the transformations (1–4) commute. For any pair of symmetries the commutativity conditions read:

$$c = \pm pcp^T \quad ; \quad p^* = \pm k^{-1}pk \quad ; \quad q = \pm pqp^\dagger \tag{10}$$

$$q^T = \pm c^\dagger q^{-1} c \quad ; \quad q^* = \pm k^{-1} q k^{\dagger\,-1} \quad ; \quad k^T c^{-1} k c^* = \pm \mathbf{1}$$

The signs $\pm$ are arbitrary; they shall correspond to different groups.

Without reality conditions.

This arises if no type $P$ and type $Q$ symmetry is imposed so that no reality condition is specified and $h$ is simply a complex matrix. We may then impose either a type $P$ or a type $C$ symmetry or both. Not all groups generated by a type $P$ and a type $C$ symmetry are distinct since, as mentioned above, the product of these symmetries is another type $C$ symmetry but with an opposite sign $\epsilon_c$. The list of inequivalent symmetry groups, together with the inequivalent choices of the sign $\epsilon_c$, is the following:

| Generators | Discrete symmetry group Defining relations | Number of classes |
|---|---|---|
| No sym | no condition | 1 |
| $P$ sym | $p^2 = 1$ | 1 |
| $C$ sym | $c^T = \pm c,\ \epsilon_c$ | 4 |
| $P$, $C$ sym | $p^2 = 1,\ c^T = \pm c,\ pcp^T = c$ | 2 |
| $P$, $C$ sym | $p^2 = 1,\ c^T = \pm \epsilon_c\, c,\ pcp^T = -c$ | 2 |

If the sign $\epsilon_c$ does not appear as an entry it means that the value of this sign is irrelevent — opposite values correspond to identical groups. The sign factors $\pm$ written explicitly are relevant — meaning that e.g. the groups generated by a type $C$ symmetry with $c^T = c$ or $c^T = -c$ are inequivalent. The equivalences among the defining relations for groups generated a type $P$ and a type $C$ symmetry are the following:

$$\begin{array}{ccc} (p^2 = 1, c^T = \pm c, c = pcp^T)_{\epsilon_c} & \cong & (p^2 = 1, c^T = \pm c, c = pcp^T)_{-\epsilon_c} \\ (p^2 = 1, c^T = \pm \epsilon_c\, c, c = -pcp^T)_{\epsilon_c} & \cong & (p^2 = 1, c^T = \pm \epsilon_c\, c, c = -pcp^T)_{-\epsilon_c} \end{array}$$

In the above table we anticipate the numbers of classes which we shall describe in the following section. They depend on the numbers of inequivalent representations of each set of defining relations.

Considering discrete groups generated by more symmetries of type $P$ or $C$ does not lead to new minimal classes. For instance, if the group is generated by two type $P$ symmetries, then their product commutes with $h$ and thus they do not define a minimal class. Similarly, suppose that we impose two type $C$ symmetries with sign $\epsilon_{c1}$ and $\epsilon_{c2}$. If $\epsilon_{c1}\epsilon_{c2} = -$, their product makes a type $P$ symmetry and the group they generate is among the ones listed above. If $\epsilon_{c1}\epsilon_{c2} = +$, their product commutes with $h$ and they do not specify a minimal class. More generally, considering more combinations of type $P$ and type $C$ symmetries does not lead to new

minimal classes as in such cases one may always define fixed matrices commuting with $h$.

With reality conditions.

This arises if we impose at least a type $Q$ or a type $K$ symmetry, but we may simultaneously also impose symmetries of other types. Again, there could be different but equivalent presentations of the same discrete group as not all of these symmetries are independent. For instance, a product of a type $P$ symmetry with a type $C$, $Q$ or $K$ symmetry is again a type $C$, $Q$ or $K$. The list of inequivalent groups generated by two or three of these symmetries is the following:

| Generators | Discrete symmetry groups Defining relations | Number of classes |
|---|---|---|
| $Q$ sym | $q = q^\dagger$ | 2 |
| $K$ sym | $kk^* = \pm 1$ | 2 |
| $P$, $Q$ sym | $p^2 = 1,\ q^2 = 1,\ q = \pm pqp^\dagger$ | 2 |
| $P$, $K$ sym | $p^2 = 1,\ kk^* = \pm 1,\ kp^* = pk$ | 2 |
| $P$, $K$ sym | $p^2 = 1,\ kk^* = 1,\ kp^* = -pk$ | 1 |
| $Q$, $C$ sym | $q = q^\dagger,\ c^T = \pm c,\ q^T = c^\dagger q^{-1} c,\ \epsilon_c$ | 8 |
| $Q$, $C$ sym | $q = q^\dagger,\ c^T = \pm c,\ q^T = -c^\dagger q^{-1} c,\ \epsilon_c$ | 4 |
| $P$, $Q$, $C$ sym | $p^2 = 1,\ q = q^\dagger,\ c^T = \pm c,\ \epsilon_c$ | |
| | $c = \epsilon_{cp}\, pcp^T,\ q = \epsilon_{pq}\, pqp^\dagger,\ q^T = \epsilon_{cq}\, c^\dagger q^{-1} c$ | 12 |

As above, the explicitly mentioned signs $\pm$ correspond to inequivalent groups. The groups with defining relations $(p^2 = 1,\ kk^* = 1,\ kp^* = -pk)$ or $(p^2 = 1,\ kk^* = -1,\ kp^* = -pk)$ are equivalent. The groups generated either by a type $Q$ and a type $K$, or by a type $C$ and a type $K$ symmetries are included in this list because the symmetries of type $C$, $Q$ or $K$ are linked by the fact the product of two of them produces a symmetry of the third type. The last cases, quoted in the last line of the above list, are made of groups generated by three symmetries, one of a type $P$ and two of type either $C$, $Q$ or $K$. Their defining relations depend on the choices of the signs $\epsilon_{cp}$, $\epsilon_{pq}$ and $\epsilon_{cq}$. The equivalences between these choices are the following:

$$\begin{array}{ccc}
(c^T = \pm c; \epsilon_{cp} = \epsilon_{pq} = \epsilon_{cq} = +)_{\epsilon_c} & \cong & (c^T = \pm c; \epsilon_{cp} = \epsilon_{pq} = \epsilon_{cq} = +)_{-\epsilon_c} \\
(c^T = \pm c; \epsilon_{cp} = \epsilon_{pq} = -\epsilon_{cq} = +)_{\epsilon_c} & \cong & (c^T = \pm c; \epsilon_{cp} = \epsilon_{pq} = -\epsilon_{cq} = +)_{-\epsilon_c} \\
(c^T = \pm c; \epsilon_{cp} = -\epsilon_{pq} = \epsilon_{cq} = +)_{\epsilon_c} & \cong & (c^T = \pm c; -\epsilon_{cp} = \epsilon_{pq} = \epsilon_{cq} = -)_{-\epsilon_c} \\
(c^T = \pm \epsilon_c\, c; -\epsilon_{cp} = \epsilon_{pq} = \epsilon_{cq} = +)_{\epsilon_c} & \cong & (c^T = \pm \epsilon_c\, c; \epsilon_{cp} = -\epsilon_{pq} = \epsilon_{cq} = -)_{-\epsilon_c} \\
(c^T = \pm \epsilon_c\, c; \epsilon_{cp} = \epsilon_{pq} = -\epsilon_{cq} = -)_{\epsilon_c} & \cong & (c^T = \pm \epsilon_c\, c; \epsilon_{cp} = \epsilon_{pq} = \epsilon_{cq} = -)_{-\epsilon_c}
\end{array}$$

Considering groups generated by more symmetries does not lead to new minimal classes since in such cases one may construct matrices commuting with $h$.

## 3. Explicit realizations of the classes.

Having determined the inequivalent groups of commuting discrete symmetries, the second step consists in finding all inequivalent representations of the defining relations of those groups. Due to the rules we choose, in particular the second one, eq.(6), we only consider representations in which all symmetries are unitarily implemented and which are not unitarily equivalent.

We shall list all these representations, adding an index $\epsilon_c$ to recall when the action (1) of the corresponding discrete group depends on $\epsilon_c$.

Without reality conditions:

If we impose no symmetry at all, the class is simply the set of complex matrices.

If we impose only a type $P$ symmetry. The matrix $p$ is unitary and square to the identity, so that it is diagonalizable with eigenvalues $\pm 1$. The solution $p = 1$ is trivial as it implies $h = 0$. Assuming for simplicity that the numbers of $+1$ and $-1$ eigenvalues are equal, we may choose a basis diagonalizing $p$:

$$(p = \sigma_z \otimes 1); \tag{11}$$

Here and below, $\sigma_z,\ \sigma_x$ and $\sigma_y$ denote the standard Pauli matrices.

If we only impose a type $C$ symmetry, $c$ is unitary and either symmetric and antisymmetric. As is well known [2], up to an appropriate gauge choice it may be presented into one of the following forms:

$$(c = 1)_{\epsilon_c}; \quad (c = i\sigma_y \otimes 1)_{\epsilon_c}; \tag{12}$$

Let us now impose a type $P$ and a type $C$ symmetry. In the basis diagonalizing $p$ with $p = \sigma_z \otimes 1$, the commutativity relation $pcp^T = \pm c$ means that either $p$ and $c$ commute or anticommute. If they commute, then $c$ is block diagonal in this basis so that $c = 1 \otimes 1$ or $c = 1 \otimes i\sigma_y$ depending whether it is symmetric or antisymmetric. If they anticommute, $c$ is block off-diagonal so that, modulo unitary change of basis, it may be reduced to $c = \sigma_x \otimes 1$ or $c = i\sigma_y \otimes 1$. However, as explained in the previous intrinsic classification, these two cases correspond the same symmetry group but with opposite signs $\epsilon_c$. So a set of inequivalent representations is:

$$\begin{aligned}(p = \sigma_z \otimes 1,\ c = 1 \otimes 1); (p = \sigma_z \otimes 1 \otimes 1,\ c = 1 \otimes i\sigma_y \otimes 1);\\ (p = \sigma_z \otimes 1,\ c = \sigma_x \otimes 1)_{\epsilon_c};\end{aligned} \tag{13}$$

Altogether there are 10 classes of non-hermitian random matrices without reality conditions. They are of course parallel to the 10 classes of hermitian random matrices [4].

With reality conditions:

Imposing only a type $Q$ symmetry, we have $q = q^\dagger$ and $qq^\dagger = 1$ since, by choice, we implement the symmetry unitarily. So $q$ is diagonalizable with eigenvalues $\pm 1$. The solution $q = 1$ is non-trivial as it simply means that $h$ is hermitian. When $q$ possesses both $+1$ and $-1$ eigenvalues we assume that they are in equal numbers, hence:

$$(q = 1); \quad (q = \sigma_z \otimes 1); \tag{14}$$

Imposing only a type $K$ symmetry, we have $kk^* = \pm 1$, $kk^\dagger = 1$, and their classification is similar to that of type $C$ symmetries. There are two cases:

$$(k = 1); \quad (k = i\sigma_y \otimes 1); \tag{15}$$

When imposing both type $P$ and $Q$ symmetries, $p$ and $q$ both square to the identity and either commute and anticommute. In the gauge in which $p = \sigma_z \otimes 1$ the possible $q$ are $1 \otimes 1$, $\sigma_z \otimes 1$ or $\sigma_x \otimes 1$, up to unitary similarity transformations preserving the form of $p$. However, the two first possibilities generate identical groups, thus the inequivalent representations are:

$$(p = \sigma_z \otimes 1,\ q = 1 \otimes 1); \quad (p = \sigma_z \otimes 1,\ q = \sigma_x \otimes 1); \tag{16}$$

Let us next impose a type $P$ and a type $K$ symmetry. With the unitarity property of $p$ and $k$, the commutativity relations between these symmetries are similar to those between type $P$ and $C$ symmetries. Thus, the classification of these representations may be borrowed from eq.(13) and we have:

$$\begin{gathered}(p = \sigma_z \otimes 1,\ k = 1 \otimes 1); (p = \sigma_z \otimes 1 \otimes 1,\ k = 1 \otimes i\sigma_y \otimes 1); \\ (p = \sigma_z \otimes 1,\ k = \sigma_x \otimes 1);\end{gathered} \tag{17}$$

When imposing simultaneously a type $Q$ and a type $C$ symmetry, their product is a symmetry of type $K$. The commutativity constraints between type $Q$ and $C$ symmetries are solved in the same way as the commutativity conditions for type $P$ and $C$ symmetries; the only difference being that $q = 1$ is now a non-trivial solution, contrary to $p = 1$. The inequivalent representations for $q$ and $c$ are:

$$\begin{aligned}(q = 1,\ c = 1)_{\epsilon_c} \quad &; \quad (q = 1 \otimes 1,\ c = i\sigma_y \otimes 1)_{\epsilon_c}; \\ (q = \sigma_z \otimes 1,\ c = 1 \otimes 1)_{\epsilon_c} \quad &; \quad (q = \sigma_z \otimes 1,\ c = 1 \otimes i\sigma_y)_{\epsilon_c} \\ (q = \sigma_z \otimes 1,\ c = i\sigma_y \otimes 1)_{\epsilon_c} \quad &; \quad (q = \sigma_z \otimes 1,\ c = \sigma_x \otimes 1)_{\epsilon_c};\end{aligned} \tag{18}$$

Finally, let us impose a type $P$ symmetry together with two among the three types of symmetries $Q$, $C$ and $K$. The commutativity constraints are solved by first choosing a gauge in which $p = \sigma_z \otimes 1$. The list of inequivalent solutions is:

$$\begin{aligned}
&(p = \sigma_z \otimes 1, \quad q = 1 \otimes 1, \quad c = 1 \otimes 1 \text{ or } c = 1 \otimes i\sigma_y); \\
&(p = \sigma_z \otimes 1, \quad q = 1 \otimes 1, \quad c = \sigma_x \otimes 1)_{\epsilon_c}; \\
&(p = \sigma_z \otimes 1, \quad q = \sigma_x \otimes 1, \quad c = 1 \otimes 1 \text{ or } c = 1 \otimes i\sigma_y \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{or } c = \sigma_x \otimes 1 \text{ or } c = \sigma_x \otimes i\sigma_y)_{\epsilon_c};
\end{aligned} \tag{19}$$

Here, each choice of $c$ corresponds to a different class.

Altogether, eqs.(11–19) give 43 classes of random non-hermitian matrices. As for the Ginibre ensembles [5] — which correspond to the classes without symmetry or with type $K$ symmetry with $k = 1$ or $k = i\sigma_y \otimes 1$ — we expect that in the large matrix size limit the density of states in each class covers a bounded domain of the complex plane with the topology of a disc.

## References


[1] This note does not describe the seminar presented by one of the authors (D.B.) at the conference these proceedings are meant for; the content of the seminar may be found in ref.[7]. Rather these notes present, with more details, results obtained later in ref.[8] as a byproduct of the classification of random Dirac fermions in two dimensions.

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